Calculus 12.5 Exercise

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Exercise (1). Determine whether each statement is true or false.

(a) Two lines parallel to a third line are parallel.
(b) Two lines perpendicular to a third line are parallel.
(c) Two planes parallel to a third plane are parallel.
(d) Two planes perpendicular to a third plane are parallel.
(e) Two lines parallel to a plane are parallel.
(f) Two lines perpendicular to a plane are parallel.
(g) Two planes parallel to a line are parallel.
(h) Two planes perpendicular to a line are parallel.
(i) Two planes either intersect or are parallel.
(j) Two lines either intersect or are parallel.
(k) A plane and a line either intersect or are parallel.

Solution.

(a) True.
(b) False. The $x$- and $y$-axes are both perpendicular to the $z$-axis, yet the $x$- and $y$-axes are not parallel.
(c) True.
(d) False. The $xy$- and $yz$-planes are not parallel, yet they are both perpendicular to the $xz$-plane.
(e) False. The $x$- and $y$-axes are not parallel, yet they are both parallel to the plane $z = 1$.
(f) True.
(g) False. The planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the $x$-axis.
(h) True.
(i) True.
(j) False. They can be skew.
(k) True.

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Find a vector equation and parametric equations for the line.

**Exercise** (5). The line through the point \((1, 0, 6)\) and perpendicular to the plane \(x + 3y + z = 5\).

**Solution.** A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as \(\mathbf{n} = (1, 3, 1)\). So, a vector equation is

\[
\mathbf{r} = (1 + t)\mathbf{i} + 3t\mathbf{j} + (6 + t)\mathbf{k}
\]

and a parametric equation is

\[
x = 1 + t, \quad y = 3t, \quad z = 6 + t.
\]

Find parametric equations and symmetric equations for the line.

**Exercise** (11). The line through \((1, -1, 1)\) and parallel to the line \(x + 2 = \frac{1}{2}y = z - 3\).

**Solution.** The line has direction \(\mathbf{v} = (1, 2, 1)\). A parametric equation is

\[
x = 1 + t, \quad y = -1 + 2t, \quad z = 1 + t
\]

and a symmetric equation is

\[
x - 1 = \frac{y + 1}{2} = z - 1.
\]

**Exercise** (15).

(a) Find symmetric equations for the line that passes through the point \((0, 2, -1)\) and is parallel to the line with parametric equations \(x = 1 + 2t, y = 3t, z = 5 - 7t\).

(b) Find the points in which the required line in part (a) intersects the coordinate planes.

**Solution.**

(a) A direction vector of the line is \(\mathbf{v} = (2, 3, -7)\). Here \(P_0 = (0, 2, -1)\), so symmetric equations for the line are

\[
\frac{x}{2} = \frac{y - 2}{3} = \frac{z + 1}{-7}.
\]

(b) The line intersects the plane when \(z = 0\), so we need \(x = \frac{-7}{2}, y = \frac{11}{2}\). Thus the point of intersection with the \(zy\)-plane is \((-\frac{7}{2}, \frac{11}{2}, 0)\). Similarly for the \(yz\)-plane, we have \((0, 2, -1)\). For the \(xz\)-plane, we have \((-\frac{7}{2}, 0, \frac{11}{2})\).

Determine whether the lines \(L_1\) and \(L_2\) are parallel, skew, or intersecting. If they intersect, find the point of intersection.

**Exercise** (19).

\[
L_1:\quad x = -6t, \quad y = 1 + 9t, \quad z = -3t
\]

\[
L_2:\quad x = 1 + 2s, \quad y = 4 - 3s, \quad z = s
\]

**Solution.** Since the direction vectors are \(\mathbf{v}_1 = (-6, 9, -3)\) and \(\mathbf{v}_2 = (2, -3, 1)\), we have so the lines are parallel.

Find an equation of the plane.

**Exercise** (29). The plane through the point \((4, -2, 3)\) and parallel to the plane \(3x - 7z = 12\).

**Solution.** Since the two planes are parallel, they will have the same normal vectors. So we can take \(\mathbf{n} = (3, 0, -7)\), and an equation of the plane is \(3(x - 4) + 0[y - (-2)] - 7(z - 3) = 0\) or

\[
3x - 7z = -9.
\]

**Exercise** (37). The plane that passes through the point \((-1, 2, 1)\) and contains the line of intersection of the planes \(x + y - z = 2\) and \(2x - y + 3z = 1\).
Solution. A direction vector for the line of intersection is
\[ \mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = (1, 1, -1) \times (2, -1, 3) = (2, -5, -3), \]
and \( \mathbf{a} \) is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection - for example, \((0, \frac{7}{2}, \frac{3}{2})\) - to the given point \((-1, 2, 1)\) in the plane. So we have another vector \((-1, -\frac{3}{2}, -\frac{1}{2})\) which is parallel to the plane. Then a normal vector to the plane is
\[ \mathbf{n} = (2, -5, -3) \times (-1, -\frac{3}{2}, -\frac{1}{2}) = (-2, 4, -8) \]
and an equation of the plane is \(-2(x + 1) + 4(y - 2) - 8(z - 1) = 0\) or
\[ x - 2y + 4z = -1. \]

Find the point at which the line intersects the given plane.

**Exercise (41).** \( x = y - 1 = 2z; 4x - y + 3z = 8 \)

**Solution.** Parametric equations for the line are
\[ x = t, \quad y = 1 + t, \quad z = \frac{1}{2} t. \]
Substituting it into the equation of the plane, we have \( t = 2 \). Thus, the point of intersection is \((2, 3, 1)\).

Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

**Exercise (49).** \( x = 4y - 2z, 8y = 1 + 2x + 4z \)

**Solution.** The normals are \( \mathbf{n}_1 = (1, -4, 2) \) and \( \mathbf{n}_2 = (2, -8, 4) \). Since \( \mathbf{n}_2 = 2\mathbf{n}_1 \), the normals, and thus the planes, are parallel.

**Exercise (59).** Find parametric equations for the line through the point \((0, 1, 2)\) that is parallel to the plane \( x + y + z = 2 \) and perpendicular to the line \( x = 1 + t, y = 1 - t, z = 2t \).

**Solution.** Two vectors which are perpendicular to the required line are the normal of the given plane, \((1, 1, 1)\), and a direction vector for the given line, \((1, -1, 2)\). So a direction vector for the required line is \( \mathbf{v} = (1, 1, 1) \times (1, -1, 2) = (3, -1, -2) \).

Thus \( \mathbf{L} \) is given by
\[ x = 3t, \quad y = 1 - t, \quad z = 2 - 2t. \]

Use the formula in Exercise 39 in Section 12.4 to find the distance from the point to a given line.

**Exercise (63).** \((1, 2, 3); x = 2 + t, \quad y = 2 - 3t, \quad z = 5t \)

**Solution.** Let \( Q = (2, 2, 0) \) and \( R = (3, -1, 5) \), points on the line corresponding to \( t = 0 \) and \( t = 1 \). Let \( P = (1, 2, 3) \) and \( \mathbf{a} = \overrightarrow{QR} = (1, -3, 5), \mathbf{b} = \overrightarrow{QP} = (-1, 0, 3). \) The distance is
\[ d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|(1, -3, 5) \times (1, 0, 3)|}{|(1, -3, 5)|} = \frac{|(-9, -8, -3)|}{|(1, -3, 5)|} \]
\[ = \frac{\sqrt{(-9)^2 + (-8)^2 + (-3)^2}}{\sqrt{1^2 + (-3)^2 + 5^2}} = \frac{\sqrt{154}}{\sqrt{35}} = \sqrt{\frac{22}{5}} \]

Find the distance between the given parallel planes.

**Exercise (67).** \( z = x + 2y + 1, 3x + 6y - 3z = 4 \)

**Solution.** Put \( y = z = 0 \) in the equation of the first plane to get the point \((-1, 0, 0)\) on the plane. Because the planes are parallel, the distance \( d \) between them is the distance from \((-1, 0, 0)\) to the second plane. By Equation 9,
\[ d = \frac{|3(-1) + 6(0) - 3(0) - 4|}{\sqrt{3^2 + 6^2 + (-3)^2}} = \frac{7}{3\sqrt{6}} = \frac{7\sqrt{6}}{18} \]