Calculus 10.3 Exercise

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Exercise (13). Find the distance between the points with polar coordinates $(1, \frac{\pi}{6})$ and $(3, \frac{3\pi}{4})$.

Solution. Since $x = r \cos \theta$, $y = r \sin \theta$, we have

$$(x_1, y_1) = (1 \cos \frac{\pi}{6}, 1 \sin \frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2}),$$

$$(x_2, y_2) = (3 \cos \frac{3\pi}{4}, 3 \sin \frac{3\pi}{4}) = (-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}).$$

Hence, the distance

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{\left(\frac{\sqrt{3}}{2} + \frac{3\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2} - \frac{3\sqrt{2}}{2}\right)^2}$$

$$= \frac{1}{2} \sqrt{40 + 6\sqrt{6} - 6\sqrt{2}}.$$

Exercise (27). For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for a curve.

(a) A line through the origin that makes an angle of $\pi/6$ with the positive $x$-axis.

(b) A vertical line through the point $(3, 3)$.

Solution.

(a) We have

$$\theta = \frac{\pi}{6}$$

directly. Since $\tan \theta = \frac{y}{x}$, we obtain

$$y = \tan \frac{\pi}{6} x = \frac{1}{\sqrt{3}} x.$$

(b) The easier description here is the Cartesian equation $x = 3$.

Exercise (31). Sketch the curve with the given polar equation

$$r = \sin \theta$$

Solution. By this polar equation, we have $r^2 = r \sin \theta$ by multiplying $r$ both sides. Since $r^2 = x^2 + y^2$ and $r \sin \theta = y$, we have

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

a circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

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Exercise (47). The figure shows the graph of \( r \) as a function of \( \theta \) in Cartesian coordinates. Use it to sketch the corresponding polar curve.

Solution. See Figure-Ex47. For \( \theta = 0, \pi, 2\pi \), \( r \) has its minimum value of about \( \frac{1}{2} \). For \( \theta = \frac{\pi}{2}, \frac{3\pi}{2} \), \( r \) attains its maximum value of 2. We see that the graph has a similar shape for \( 0 \leq \theta \leq \pi \) and \( \pi \leq \theta \leq 2\pi \).

Exercise (55). Find the slope of the tangent line to the given polar curve at the point specified by the value of \( \theta \).

\[ r = 2 \sin \theta, \quad \theta = \pi/6. \]

Solution. For the polar equation \( r = 2 \sin \theta \), we have

\[
x = r \cos \theta = 2 \sin \theta \cos \theta = \sin 2\theta \\
y = r \sin \theta = 2 \sin^2 \theta
\]

and slope

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2 \sin \theta \cos \theta}{2 \cdot \cos 2\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta.
\]

When \( \theta = \pi/6 \), the slope

\[
\frac{dy}{dx} = \tan \frac{\pi}{3} = \sqrt{3}.
\]

Exercise (65). Find the points on the given curve where the tangent line is horizontal or vertical.

\[ r = \cos 2\theta. \]

Solution. For the polar equation \( r = \cos 2\theta \), we have

\[
x = r \cos \theta = \cos 2\theta \cos \theta \\
y = r \sin \theta = \cos 2\theta \sin \theta
\]
and

\[
\frac{dx}{d\theta} = -2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta \\
= -4 \sin \theta \cos^2 \theta - (2 \cos^2 \theta - 1) \sin \theta \\
= \sin \theta (1 - 6 \cos^2 \theta)
\]

\[
\frac{dy}{d\theta} = -2 \sin 2\theta \sin \theta + \cos 2\theta \cos \theta \\
= -4 \sin^2 \theta \cos \theta + \cos^3 \theta - \sin^2 \theta \cos \theta \\
= \cos \theta (\cos^2 \theta - 5 \sin^2 \theta) \\
= \cos \theta (1 - 6 \sin^2 \theta)
\]

To find the horizontal tangent line, we consider those \( \theta \) such that \( \frac{dy}{d\theta} = 0 \). Hence, \( \cos \theta = 0 \) or \( \sin \theta = \pm \frac{1}{\sqrt{6}} \), that is, \( \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha, \pi - \alpha, \pi + \alpha, 2\pi - \alpha \) where \( \alpha = \sin^{-1} \frac{1}{\sqrt{6}} \). So the tangent line is horizontal at

\((-1, \frac{\pi}{2}), (-1, \frac{3\pi}{2}), (\frac{2}{3}, \alpha), (\frac{2}{3}, \pi - \alpha), (\frac{2}{3}, \pi + \alpha), (\frac{2}{3}, 2\pi - \alpha)\)

in polar coordinates.

It is similar to find the vertical tangent line, For \( \frac{dx}{d\theta} = 0 \), we have \( \sin \theta = 0 \) or \( \cos \theta = \pm \frac{1}{\sqrt{6}} \), that is, \( \theta = 0, \pi, \beta, \pi - \beta, \pi + \beta, 2\pi - \beta \) where \( \beta = \cos^{-1} \frac{1}{\sqrt{6}} \). So the tangent line is vertical at

\((1, 0), (1, \pi), (-\frac{2}{3}, \beta), (-\frac{2}{3}, \pi - \beta), (-\frac{2}{3}, \pi + \beta), (-\frac{2}{3}, 2\pi - \beta)\)

in polar coordinates.